J. Fluid Mech. (1965), *vol.* 23, *part* 3, *pp.* 611–623
Printed in Great Britain

Laminar boundary layer on an impulsively started rotating sphere

By EDWARD R. BENTON
National Center for Atmospheric Research, Boulder, Colorado

(Received 24 August 1964)

Viscous, incompressible, axially symmetric flow about an impulsively started rotating sphere is studied in terms of non-steady, partially linearized Navier–Stokes equations. The non-linear centripetal acceleration is included in full, but the other non-linear terms are neglected because of the restriction in interest to the case of large (but subcritical) Reynolds or Taylor numbers, $a^2\Omega/\nu$. Approximate closed-form solutions for $u(r,\theta,t)$, $v(r,\theta,t)$, $w(r,\theta,t)$ are found which satisfy all relevant boundary and initial conditions. The linearization approximation is checked for consistency and a restriction on Ωt is found. The velocity profiles, in the range of validity, are shown to be approximately similar in time, so their shapes may be qualitatively correct for larger values of Ωt. Some comparison with existing steady-state theories is given and the boundary-layer displacement thickness and viscous torque on the sphere are calculated.

1. Introduction

The viscous, incompressible flow induced by a sphere rotating in an infinite undisturbed fluid environment is of intrinsic fluid mechanical and engineering interest and also has some features in common with problems in meteorology and astrophysics. There is probably greater meteorological interest in the non-axially symmetric case, but here we consider only flows which are independent of azimuth or longitude. The present interest arose in connexion with a study of the undesirable rotations of high altitude balloons which currently find use as platforms for astronomical observations.

There is ample justification for investigating the impulsively started initial-value problem rather than the more pertinent steady-state flow. First, the two problems are expected to be somewhat related since the instantaneous velocity profiles should be qualitatively the same shape as the ultimate steady-state ones; in fact, an interesting result of the present non-steady analysis is that within the range of validity of the present solutions, the velocity profiles are nearly similar in time. One can then raise the interesting question as to whether or not this similarity extends indefinitely in time, in which case, the analysis used here would be useful in predicting the steady-state solution, apart from scale and amplitude factors. Still greater motivation is the fact that the non-steady problem admits greater mathematical simplification (at least for sufficiently small time) than the essentially non-linear steady-state problem. As a result, the disadvantages inherent in the boundary-layer approximation for this problem, discussed below, can be largely avoided.

In 1845, Stokes gave an accurate physical description of the steady-state flow pattern, pointing out that it is fully three-dimensional. In non-rotating spherical co-ordinates, r, θ, ϕ (with velocity components, u, v, w, respectively) the viscous no-slip boundary condition produces a circumferential or zonal component of velocity, w, referred to here as the primary flow. The resulting centrifugal forces induce a secondary flow in planes containing the axis of rotation. It consists of a radial outflow, $u > 0$, near the equator, and by continuity, there must be a radial inflow, $u < 0$, near the poles, together with a meridional flow, v, parallel to lines of longitude at intermediate latitudes. In engineering terms, the sphere acts like a centrifugal fan; meteorologically, there is a centrifugally driven Hadley type circulation.

Modern research on the steady-state problem by Howarth (1951), Nigam (1954), Kobashi (1957), Stewartson (1958), and Kreith, Roberts, Sullivan & Sinha (1963) contains considerable disagreement among authors, much of which can be traced to the method of approximation used. Howarth (1951) introduced the boundary-layer equations for a rotating sphere and showed that, in the neighbourhood of the poles ($\theta = 0, \pi$), they reduce to the well-known von Kármán equations for an infinite rotating disk. Presumably, therefore, the solution to the sphere problem must approach that of the disk as $\theta \rightarrow 0$ or π, but as elaborated below, this conclusion is not fully justified because the two problems have different boundary conditions at infinity. Nevertheless, Howarth used the rotating disk analogy to make an appropriate choice of functional dependence for v, w and obtained an approximate solution to the boundary-layer momentum integral equations by the Kármán–Pohlhausen method. Subsequent theoretical analyses have also relied on the boundary-layer equations and the Kármán–Pohlhausen method. In spite of the generally great usefulness of these approximations, there are several important disadvantages: the Navier–Stokes equations are changed in mathematical type, only polynomial approximations are made within a finite boundary layer, the order of the system is reduced by 1, and the full continuity equation is not satisfied.

More specifically, the Navier–Stokes equations are elliptic in the co-ordinates whereas the boundary-layer equations are parabolic. Fully aware of this, Howarth indicated that the latter equations cannot adequately describe the interesting interaction near the equator where the secondary flow is expected to consist of two colliding boundary layers. Thus, Howarth's solution gives radial inflow to the boundary layer everywhere, no outflow being predicted near the equator. In 1954, Nigam disagreed with Howarth's assessment of the parabolic failure of the boundary-layer equations, but his argument was subsequently disputed by Stewartson (1958) and recently by Kreith *et al.* (1963). Nigam introduced the following expansion for the velocity components (where $u = v_r$, $v = v_\theta$, $w = v_\phi$).

$$\left.\begin{aligned} u &= \tfrac{1}{2}(\nu\Omega)^{\frac{1}{2}}(3\cos^2\theta - 1)[F_1 + \sin^2\theta F_3 + \sin^4\theta F_5 + \ldots], \\ v &= a\Omega\cos\theta\sin\theta[G_1 + \sin^2\theta G_3 + \sin^4\theta G_5 + \ldots], \\ w &= a\Omega\sin\theta[H_1 + \sin^2\theta H_3 + \sin^4\theta H_5 + \ldots]. \end{aligned}\right\} \quad (1)$$

Here a and Ω are the sphere's radius and angular velocity, respectively, and the

F, G, H functions give the radial dependence. These first nine functions were determined from the boundary-layer equations by the Kármán–Pohlhausen method. The present analysis shows that, at least to first order, Nigam's trigonometric dependence is logical. Furthermore, equations (1) provide for both inflow and outflow from the boundary layer, as demanded by continuity. The critical co-latitude, θ_c, where $u = 0$, is given by $3\cos^2\theta_c = 1$, or $\theta_c = 54{\cdot}7°$. This angle was also determined experimentally by Kobashi (1957) who found $\theta_c = 54{\cdot}5°$. Such close agreement must, of course, be regarded as partly fortuitous since, as noted by Kreith *et al.* (1963), Kobashi deduced the very small radial velocity component indirectly from hot-wire measurements of the total velocity vector.

A second shortcoming of the previous studies rests in the Kármán–Pohlhausen method. The point at infinity where the velocity vanishes is moved in to a finite distance from the boundary and then the relevant functions are represented by polynomials in the finite boundary layer. There is no *a priori* reason to suppose that all velocity components vanish at the same distance from the sphere. Moreover, the finite depth approximation coupled with Nigam's expansion, equations (1), leads to a practically constant boundary-layer thickness over the entire sphere. Accordingly, Nigam's solution has been doubted by Kobashi (1957) and Kreith *et al.* (1963) both of whom measured values of δ which varied with θ. However, any measurements of δ must be viewed with caution since the radial distance at which the circumferential velocity decreases to 1 or 2 % of the surface speed is not very precisely located and is highly sensitive to experimental error. In fact, Kobashi's and Kreith's measurements only agree within about 25 % on this point.

The last objections to the boundary-layer approximation for this problem are the most serious since they lead to results which are not even qualitatively correct far out in the boundary layer. Within that approximation, the radial pressure gradient is small so the radial equation of motion is neglected. This lowers the order of the system by 1 so one boundary condition cannot be satisfied. In Howarth's and Nigam's treatments, the radial velocity tends to a finite non-zero limit at $r = a+\delta$. As a by-product, because of the boundary-layer approximation for the continuity equation, their meridional velocity fields, v, are directed toward the equator everywhere, which violates continuity. There must be a poleward flow sufficiently far out to feed the secondary circulation; indeed, one of the questions of interest is the distance from the sphere at which v changes sign. A non-zero radial flow at $r = a+\delta$ may appear justified near the axis of rotation in view of Howarth's proof that the boundary-layer equations in that region reduce to those of the von Kármán rotating disk. However, for the disk problem a non-zero flow towards the boundary is required far away only because the disk, being of infinite radius, forces a radial outflow at infinity. The sphere has finite radius, a, so near the poles the solution can only approach that corresponding to a finite disk. In this paper we impose the physically realistic boundary conditions that all velocity components vanish as $r \to \infty$.

2. Formulation

Consider a sphere of radius a immersed in an infinite expanse of homogeneous viscous fluid initially at rest. At time $t = 0$, an impulsive angular acceleration is applied to the sphere so that its angular velocity instantaneously increases to a constant value Ω, so chosen that the Reynolds or Taylor number, $a^2\Omega/\nu$, is much larger than 1. The continuity and Navier–Stokes equations, in non-rotating spherical co-ordinates r, θ, ϕ, with $\partial/\partial\phi \equiv 0$ because of axial symmetry, are

$$r^{-2}(r^2u)_r + (r\sin\theta)^{-1}(v\sin\theta)_\theta = 0, \tag{2}$$

$$\begin{aligned} &u_t + uu_r + r^{-1}u_\theta v - r^{-1}(v^2+w^2) \\ &\quad = \rho^{-1}p_r + \nu[r^{-2}(r^2u_r)_r + (r^2\sin\theta)^{-1}(u_\theta\sin\theta)_\theta - 2r^{-2}(u + v_\theta + v\cot\theta)], \end{aligned} \tag{3}$$

$$\begin{aligned} &v_t + uv_r + r^{-1}vv_\theta + r^{-1}uv - r^{-1}w^2\cot\theta \\ &\quad = (\rho r)^{-1}p_\theta + \nu[r^{-2}(r^2v_r)_r + (r^2\sin\theta)^{-1}(v_\theta\sin\theta)_\theta + 2r^{-2}u_\theta - (r\sin\theta)^{-2}v], \end{aligned} \tag{4}$$

$$\begin{aligned} &w_t + uw_r + r^{-1}vw_\theta + r^{-1}uw + r^{-1}vw\cot\theta \\ &\quad = \nu[r^{-2}(r^2w_r)_r + (r^2\sin\theta)^{-1}(w_\theta\sin\theta)_\theta - (r\sin\theta)^{-2}w]. \end{aligned} \tag{5}$$

The initial condition, no-slip boundary condition and vanishing of the velocity at infinity (as discussed in the introduction) take the form

$$\left.\begin{array}{ll} \text{at} & t = 0\colon u = v = w = 0, \\ \text{at} & r = a\colon u = v = 0,\ w = a\Omega\sin\theta, \\ \text{as} & r \to \infty\colon u, v, w \to 0. \end{array}\right\} \tag{6}$$

The method of simplification used here is standard for impulsively started flows (e.g. Schlichting 1960). It has been used by Carrier & DiPrima (1956) who studied the small rotational oscillations of a sphere. In fact, they also suggested using it for the present initial value problem. Our notation is similar to theirs, although the solutions, of course, are different. In the early stages of motion, i.e. when Ωt is suitably restricted, the non-steady accelerations in equations (3)–(5) are large and since the boundary layer is initially thin (i.e. δ is small compared with a), the viscous forces will be large, but the convective acceleration will be relatively small. Stated another way, we assume, for sufficiently small Ωt, that the secondary flow is small relative to the primary flow: u, v small compared to w. This argument is made more quantitative by noting that since w decays from $a\Omega\sin\theta$ at $r = a$ toward 0 at $r = a+\delta$, then w is of order $a\Omega$ within the boundary layer. Consequently, the centripetal accelerations which induce the secondary flow are of order $a\Omega^2$, so that for small t, u and v are of order $a\Omega^2 t$. The ratio of u or v to w is then of order $\epsilon = \Omega t$. One could now proceed formally to expand u, v, w in powers of ϵ and show that first approximations are determined by equations with all non-linear terms except $r^{-1}w^2$ neglected. This was done by Carrier & DiPrima (1956) and is not repeated here. In any case, the linearization hypothesis is checked *post facto* and an upper limit on Ωt is found.

If the influence of the secondary flow on the primary flow is first neglected, then equation (5) becomes

$$w_t = \nu[r^{-2}(r^2w_r)_r + (r^2\sin\theta)^{-1}(w_\theta\sin\theta)_\theta - (r\sin\theta)^{-2}w]. \tag{7}$$

Once w is found from this linear equation, it is used as a forcing function to drive a secondary flow. A stream function is introduced to satisfy equation (2)

$$u \equiv -(r\sin\theta)^{-1}(\psi\sin\theta)_\theta, \quad v \equiv r^{-1}(r\psi)_r. \tag{8}$$

Equations (3) and (4) are now partially linearized by neglecting terms quadratic in u, v compared to the linear terms in u, v. However, the important non-linear centripetal acceleration involving w^2/r is retained in full. Equations (3) and (4) are then combined to eliminate the pressure in the usual way by considering the ϕ component of the vorticity which, in view of equation (8) leads to the following single equation for $\psi(r,\theta,t)$

$$\mathscr{L}(\psi_t) + \nu\mathscr{L}^2(\psi) = r^{-2}(w^2)_\theta - r^{-1}\cot\theta(w^2)_r, \tag{9}$$

where $\mathscr{L}$ is the linear differential operator

$$\mathscr{L}(h) \equiv -r^{-1}\{(rh)_{rr} + r^{-1}[(\sin\theta)^{-1}(h\sin\theta)_\theta]_\theta\}. \tag{10}$$

3. The primary flow w

(i) *Solution by Laplace transform*

Appropriate non-dimensional variables used throughout are

$$x = r/a, \quad \tau = \nu t/a^2 = \Omega t/R, \quad R = a^2\Omega/\nu. \tag{11}$$

Since the boundary-layer thickness is expected to be of order $(\nu t)^{\frac{1}{2}}$, the non-dimensional time τ is essentially the ratio of δ^2 to a^2. It is also the ratio of the sphere's angular rotation Ωt to the Reynolds or Taylor number, $R = a^2\Omega/\nu$. Introduction of the non-dimensional primary flow

$$w(r,\theta,t) = a\Omega\sin\theta f(x,\tau), \tag{12}$$

eliminates the trigonometric dependence from equation (7), leaving

$$f_\tau - f_{xx} - 2x^{-1}f_x + 2x^{-2}f = 0, \tag{13}$$

with boundary and initial conditions

$$f(x,0) = 0, \quad f(1,\tau) = 1, \quad f(\infty,\tau) = 0. \tag{14}$$

The boundary-layer approximation to equation (13), in which the curvature of the boundary is neglected, would consist of just the first two terms, but it is not necessary to introduce that simplification at this point because the exact solution of this linear parabolic boundary-value problem can be found by introducing the Laplace transform with respect to τ

$$\bar{f}(x,s) \equiv \int_0^\infty e^{-s\tau} f(x,\tau)\,d\tau. \tag{15}$$

The appropriate solution of the resulting ordinary differential equation which satisfies equations (14) is

$$\bar{f}(x,s) = \frac{1+xs^{\frac{1}{2}}}{x^2 s(1+s^{\frac{1}{2}})}\,e^{-(x-1)s^{\frac{1}{2}}}. \tag{16}$$

The inversion is conventional (cf. Erdélyi 1954), with the result that

$$f(x,\tau) = x^{-2}\operatorname{erfc}\left(\frac{x-1}{2\tau^{\frac{1}{2}}}\right) + x^{-2}(x-1)\,e^{x-1+\tau}\operatorname{erfc}\left(\frac{x-1}{2\tau^{\frac{1}{2}}} + \tau^{\frac{1}{2}}\right), \tag{17}$$

where
$$\operatorname{erfc}\alpha \equiv 2\pi^{-\frac{1}{2}}\int_{\alpha}^{\infty} e^{-y^2}\,dy.$$

This solution is plotted in figure 1, for several values of τ. It shows that for sufficiently small τ, i.e. for values of τ up to about 10^{-3}, the viscous effects are confined to a narrow region adjacent to the surface. Thus, the solution for w is of boundary-layer type, for τ up to about 10^{-3}, despite the linearization of the Navier–Stokes equation for w. This is because, as discussed by Carrier (1953), boundary-layer behaviour is dependent on the smallness of the coefficient of the most highly differentiated terms rather than on non-linearity in the differential equation.

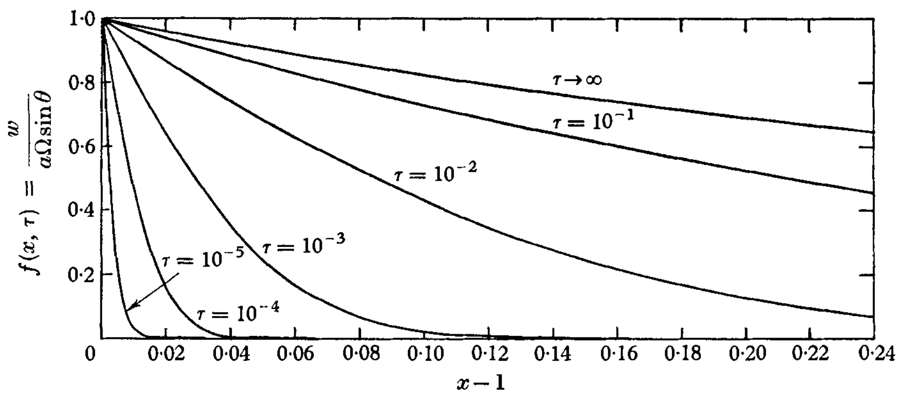

FIGURE 1. The primary (zonal) flow field as a function of radial distance for various times.

(ii) *Boundary-layer thickness*

If δ is defined as the distance from the surface of the sphere at which

$$w/a\Omega\sin\theta = 0{\cdot}02,$$

then for $0 \leqslant \tau \leqslant 10^{-2}$,
$$\delta \doteq 3{\cdot}3a\tau^{\frac{1}{2}} = 3{\cdot}3(\nu t)^{\frac{1}{2}}. \tag{18}$$

Since the linearization approximation is only valid if δ/a is small, we again see that interest must be restricted to values of τ up to about 10^{-3}, at which time $\delta/a \doteq 0{\cdot}10$.

A more precise definition of the depth of the boundary layer is the boundary-layer displacement thickness, defined here as

$$\delta_1 = (a\Omega\sin\theta)^{-1}\int_a^{\infty} w\,dr = a\int_1^{\infty} f(x,\tau)\,dx = a(1-e^{\tau}\operatorname{erfc}\tau^{\frac{1}{2}}), \tag{19}$$

where use has been made of the solution in equation (17). The leading term in an expansion of $\delta_1(\tau)$ in powers of $\tau^{\frac{1}{2}}$ is

$$\delta_1(\tau) \doteq 2\pi^{-\frac{1}{2}}a\tau^{\frac{1}{2}} = 1{\cdot}13(\nu t)^{\frac{1}{2}}. \tag{20}$$

(iii) *Viscous torque*

The viscous torque which must be overcome to maintain the rotation is the product of shear stress and moment arm, integrated over the area of the sphere

$$\begin{aligned} T &= \int_0^{2\pi}\int_0^{\pi} \mu a^4 \sin^2\theta \left[\frac{\partial}{\partial r}(r^{-1}w)\right]_{r=a} d\theta\, d\phi \\ &= \frac{8\pi}{3}\mu a^3 \Omega[(x^{-1}f)_x]|_{x=1} \\ &= -8\pi\mu a^3\Omega[1+\tfrac{1}{2}(\pi\tau)^{-\frac{1}{2}} - \tfrac{1}{3}e^{\tau}\operatorname{erfc}\tau^{\frac{1}{2}}], \end{aligned} \tag{21}$$

the minus sign indicating that the torque opposes the motion. Clearly, during the transient where the present analysis is valid, the second term dominates, so

$$\begin{aligned} T &\doteq -\tfrac{8}{3}\pi^{\frac{1}{2}}\mu a^3\Omega\tau^{-\frac{1}{2}} = -\tfrac{8}{3}\pi^{\frac{1}{2}}(\rho\mu)^{\frac{1}{2}}a^4\Omega^{\frac{3}{2}}t^{-\frac{1}{2}} \\ &\doteq -\tfrac{16}{3}\mu a^3\Omega(a/\delta_1), \end{aligned} \tag{22}$$

the last statement following from equation (20). Equation (22) shows that the basic variation with angular velocity is non-linear, the functional dependence being the same as that for a rotating disk (Schlichting 1960). Also, the torque varies inversely with boundary-layer displacement thickness.

(iv) *Primary flow as a similar solution*

The complicated function in equation (17) is greatly simplified conceptually by observing that it is very nearly similar in time. That is, the primary flow velocity profile does not change shape very much in time, only scale, so the several curves in figure 1 can be made nearly congruent. The most obvious scale factor is the boundary-layer thickness, but because it cannot be defined precisely, we choose instead the displacement thickness, δ_1, and define the similarity variable by

$$\eta \equiv \frac{r-a}{\pi^{\frac{1}{2}}\delta_1(\tau)} = \frac{x-1}{\pi^{\frac{1}{2}}(1-e^{\tau}\operatorname{erfc}\tau^{\frac{1}{2}})} \doteq \frac{x-1}{2\tau^{\frac{1}{2}}}, \tag{23}$$

the last expression following from equation (20). At the outer edge of the boundary layer as defined by equation (18), η has the value 1·65. In terms of η, $f(x,\tau)$, is very closely approximated by

$$F(\eta) = \operatorname{erfc}\eta \doteq f(x,\tau). \tag{24}$$

The absolute error, $F(\eta)-f(x,\tau)$, never exceeds 0·0042 for values of τ up to 10^{-3} (the upper limit for validity of the present analysis). However, the percentage error does become large far out in the boundary layer. It is worth pointing out that the approximate similar solution in equation (24) is identical to that for an impulsively started flat plate (i.e. the Rayleigh problem). It is also essentially what would have been obtained from the boundary-layer approximation to equation (13).

Once the approximate similarity in time of the primary flow velocity profile has been established, it is intriguing to compare its shape with that of the steady-state profile, to see how much the two differ. If the two have the same shape, then apparently the neglected non-linear terms have had little effect on similarity. If such a conclusion were true, then the essentially non-linear steady-state profile could be predicted from the non-steady but linearized analysis used here. Of course, the present analysis cannot yield accurate values of the steady-state boundary-layer thickness, so the scale factor would remain undetermined.

Howarth's steady-state solution is given by the simple cubic

$$\frac{w}{a\Omega\sin\theta} = 1-\frac{3}{2}\left(\frac{r-a}{\delta}\right)+\frac{1}{2}\left(\frac{r-a}{\delta}\right)^3. \tag{25}$$

This same cubic is the leading term in Nigam's solution (H_1 in equation (1)). Integration of equation (25) with respect to r from a to $a+\delta$ shows that $\delta_1 = \frac{3}{8}\delta$. Therefore, in terms of our similarity variable, $\eta = (r-a)/\pi^{\frac{1}{2}}\delta_1$, Howarth's solution is

$$\frac{w}{a\Omega\sin\theta} = 1-\frac{9\pi^{\frac{1}{2}}}{16}\eta+\frac{27\pi^{\frac{3}{2}}}{1024}\eta^3. \tag{26}$$

This equation is compared with the non-steady similar solution, equation (24), in figure 2. The agreement is rather close, considering the approximations inherent in Howarth's analysis and those made here. This suggests that as time proceeds beyond the values for which the present analysis is valid, the decreasing importance of the non-steady term and the increasing importance of the non-linear terms may partially compensate each other, so that the shape of the primary flow velocity profile is relatively unaffected.

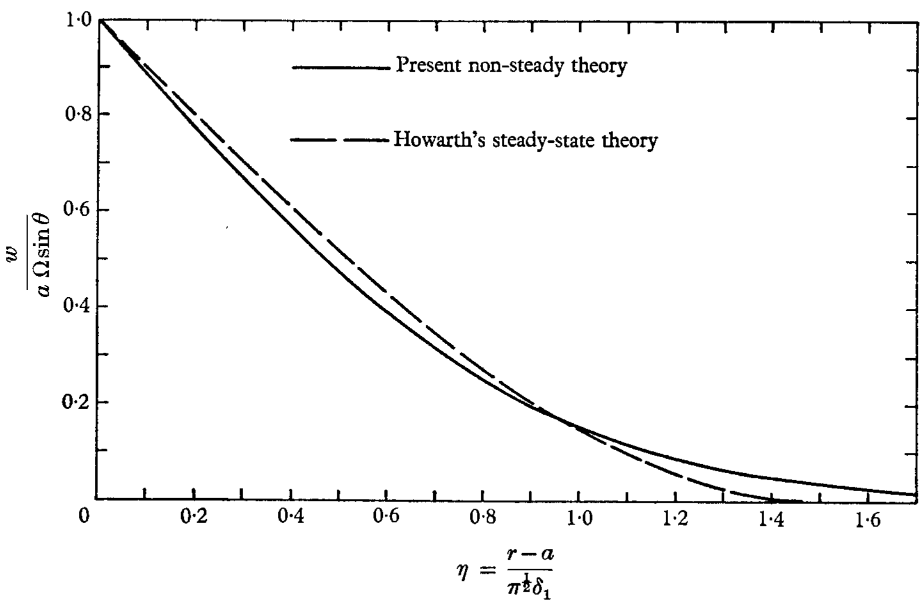

FIGURE 2. The primary (zonal) flow field as a similar solution and comparison with Howarth's steady-state boundary-layer solution.

4. The secondary flow, *u*, *v*

The starting point for finding the secondary flow is equation (9) for the stream function. This linear, fourth-order, non-homogeneous equation is first non-dimensionalized and the trigonometric dependence removed by introducing a dimensionless stream function, $g(x,\tau)$,

$$\psi(r,\theta,t) = (a^4\Omega^2/\nu)\sin 2\theta\, g(x,\tau). \tag{27}$$

In view of equation (8), the radial and meridional velocity components become

$$u = -2(a^3\Omega^2/\nu)(3\cos^2\theta-1)\,x^{-1}g, \tag{28}$$

$$v = (a^3\Omega^2/\nu)\sin 2\theta(g_x+x^{-1}g). \tag{29}$$

Comparison with equation (1) shows that Nigam's trigonometric dependence is indeed logical. Substitution of equations (11), (12) and (27) into equation (9) leads, after considerable algebra, to

$$g_{xxxx}+4x^{-1}g_{xxx}-12x^{-2}g_{xx}+24x^{-4}g-g_{\tau xx}-2x^{-1}g_{\tau x}+6x^{-2}g_\tau = x^{-2}f^2-x^{-1}ff_x. \tag{30}$$

It is remarkable that a $\sin 2\theta$ dependence in ψ completely eliminates θ from equation (9). The initial and boundary conditions are

$$g(x,0) = g(1,\tau) = g_x(1,\tau) = g(\infty,\tau) = g_x(\infty,\tau) = 0. \tag{31}$$

A convenient first integral of equation (30) can be found by first dividing by x, then integrating with respect to x, and finally multiplying by x with the result that

$$g_{xxx}+5x^{-1}g_{xx}-2x^{-2}g_x-6x^{-3}g-g_{\tau x}-3x^{-1}g_\tau = -\tfrac{1}{2}x^{-1}f^2. \tag{32}$$

The boundary conditions as $x\to\infty$ have been used to evaluate the constant of integration as 0. This last equation is still difficult to solve, primarily because the forcing function is non-linear and extremely complicated. We could, of course, make the boundary-layer approximation by replacing the left-hand side by $g_{xxx}-g_{\tau x}$, but it would then be impossible to satisfy the boundary conditions at infinity (equation (31)). Furthermore, somewhat milder approximations can be introduced which still make the equation simple enough to solve easily. In particular, use of the approximate similarity of the primary flow, f, leads to great simplification. Apart from the factor x^{-1} (which departs only very slightly from the constant value 1 throughout the boundary layer) the forcing function is very nearly similar in time. Therefore, we look for a solution of equation (32) which is, apart from powers of x, also similar in time. We introduce a similar stream function, $G(\eta)$,

$$g(x,\tau) = x^{-3}d_1^3\,G(\eta), \tag{33}$$

where $d_1 = \delta_1(\tau)/a$ is the non-dimensional displacement thickness. Here the factor x^{-3} simplifies the resulting equation and guarantees the satisfaction of boundary conditions at infinity. The factor d_1^3 essentially gives the amplitude of the stream function; in view of equation (20) it could also be approximated by $\tau^{\frac{3}{2}}$ but we prefer to make the solution appear to be independent of time so that the extent of the similarity can be checked. With this substitution, equation (32) becomes

$$G''' + (\pi d_1 d_1' \eta - 4\pi^{\frac{1}{2}} x^{-1} d_1) G'' - (2\pi d_1 d_1' - 4\pi x^{-2} d_1^2) G' = -\tfrac{1}{2}\pi^{\frac{3}{2}} x^2 f^2, \tag{34}$$

where $G' = dG/d\eta$, $d_1' = dd_1/d\tau$. The next step is to approximate the coefficients and the forcing function by simpler forms. From equation (19), an expansion of $d_1(\tau)$ in powers of $\tau^{\frac{1}{2}}$ gives

$$\left.\begin{aligned} d_1(\tau) &= 2\pi^{-\frac{1}{2}}\tau^{\frac{1}{2}} - \tau + O(\tau^{\frac{3}{2}}), \\ d_1 d_1' &= 2\pi^{-1} - 3\pi^{-\frac{1}{2}}\tau^{\frac{1}{2}} + O(\tau). \end{aligned}\right\} \tag{35}$$

Consequently, the coefficient of G'' in equation (34) is

$$2\eta[1 - \tfrac{3}{2}\pi^{\frac{1}{2}}\tau^{\frac{1}{2}} + O(\tau)] - 4x^{-1}\tau^{\frac{1}{2}}[2 - \pi^{\frac{1}{2}}\tau^{\frac{1}{2}} + O(\tau)]. \tag{36}$$

In the region of interest, η ranges from 0 at the boundary to 1·65 at the edge of boundary layer, τ is less than or equal to about 10^{-3}, and $x = r/a$ is very nearly 1. Consequently, the dominant term in equation (36) is 2η. Similarly, the coefficient of G' in equation (34) becomes

$$-4[1 - \tfrac{3}{2}\pi^{\frac{1}{2}}\tau^{\frac{1}{2}} + O(\tau)] + 16x^{-2}\tau[1 + O(\tau^{\frac{1}{2}})] \doteq -4. \tag{37}$$

The final simplification required is to approximate the correct forcing function, $-\frac{1}{2}\pi^{\frac{3}{2}}x^2 f^2$, by $-\frac{1}{2}\pi^{\frac{3}{2}}F^2$ based on equation (24) and the replacement of x by 1. Therefore, equation (34), is approximated by the simpler equation

$$G''' + 2\eta G'' - 4G' = -\tfrac{1}{2}\pi^{\frac{3}{2}} \operatorname{erfc}^2 \eta. \tag{38}$$

Essentially this same equation has been solved by Nigam (1951) in connexion with the impulsively started von Kármán rotating disk. The required solution is

$$\begin{aligned} G(\eta) = \tfrac{1}{12}\{&\pi^{\frac{1}{2}}[(3\eta + 2\eta^3)\operatorname{erfc}\eta - 2\pi^{-\frac{1}{2}}(1+\eta^2)\,e^{-\eta^2}] \\ &- \pi^{\frac{3}{2}}\eta(\pi^{-\frac{1}{2}}e^{-\eta^2} - \eta\operatorname{erfc}\eta)^2 - \pi e^{-\eta^2}\operatorname{erfc}\eta \\ &+ 2^{\frac{1}{2}}\pi\operatorname{erfc}2^{\frac{1}{2}}\eta + 2 - (2^{\frac{1}{2}} - 1)\,\pi\}. \end{aligned} \tag{39}$$

Since G' is needed to evaluate v, we list it also for reference

$$\begin{aligned} G'(\eta) = &\tfrac{1}{12}\pi^{\frac{1}{2}}[(3 + 6\eta^2)\operatorname{erfc}\eta - 6\pi^{-\frac{1}{2}}\eta\, e^{-\eta^2}] \\ &- \tfrac{1}{4}\pi^{\frac{3}{2}}(\pi^{-\frac{1}{2}}\,e^{-\eta^2} - \eta\operatorname{erfc}\eta)^2. \end{aligned} \tag{40}$$

The radial and meridional velocity components now have the form (from equations (28), (29), (33))

$$u = -2(a^3\Omega^2/\nu)\,(3\cos^2\theta - 1)x^{-4}d_1^3 G, \tag{41}$$

$$v = (a^3\Omega^2/\nu)\sin 2\theta(\pi^{-\frac{1}{2}}x^{-3}d_1^2 G' - 2x^{-4}d_1^3 G). \tag{42}$$

5. Results and discussion

Figure 3 shows G and G' plotted as functions of η. It should be noted that even though G has a non-zero asymptotic value for large η, the velocity components u, v tend to zero as $x \to \infty$, as required. In figures 4 and 5 the radial dependence of u and v are displayed for three values of τ, and it is seen that the shapes of these

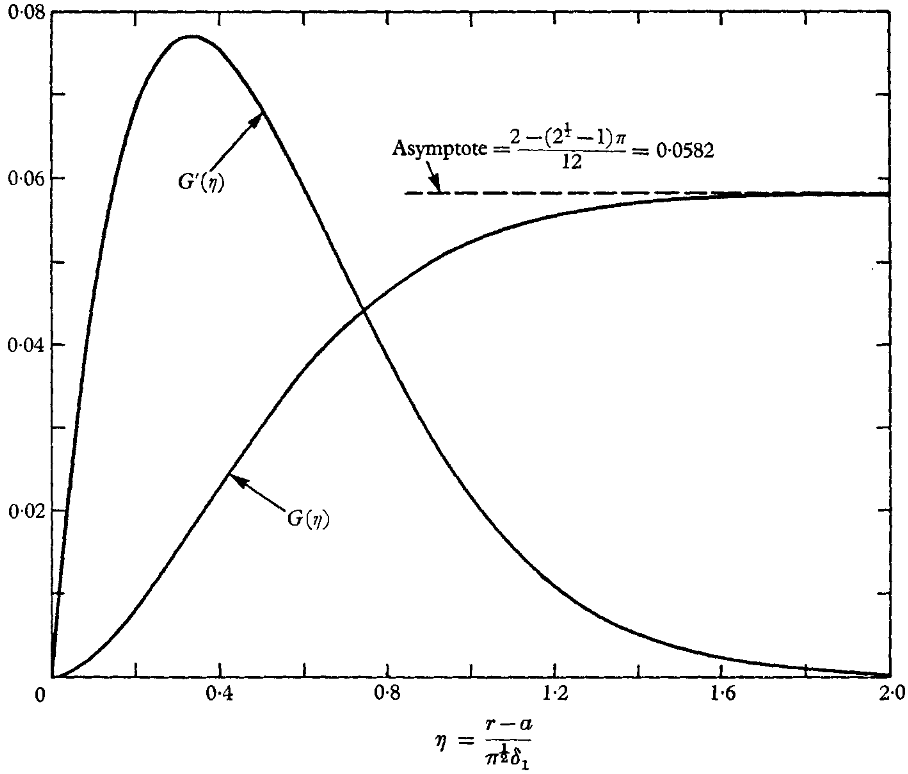

FIGURE 3. The non-dimensional stream function and its derivative as functions of the similarity variable.

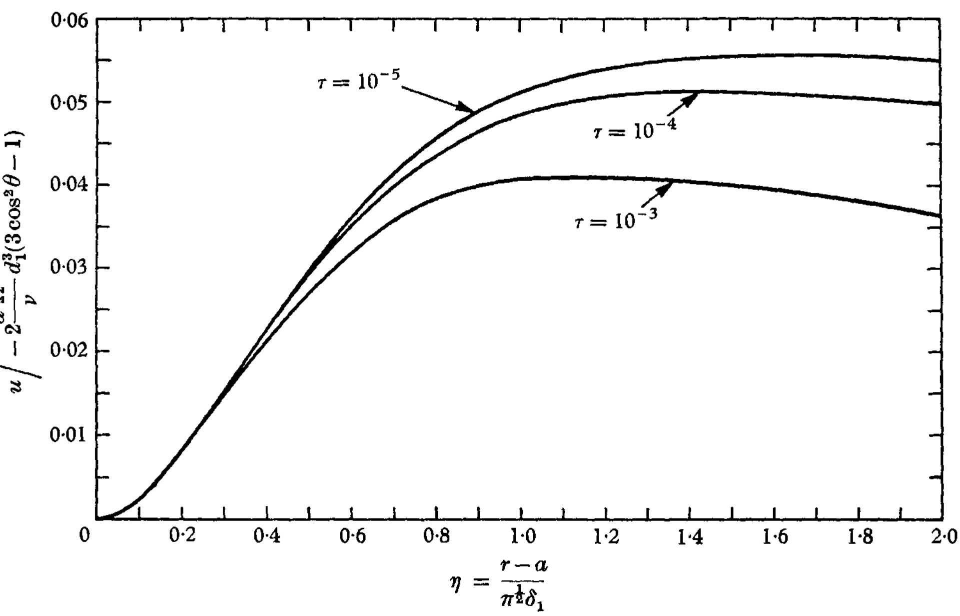

FIGURE 4. The radial dependence of the radial velocity component for three values of time.

velocity profiles are qualitatively correct in the sense that u has a maximum and then decays far out and v changes sign, as demanded by continuity. These last two features are not present in the solutions of previously cited authors. On the other hand, it must be admitted that the present solutions cannot be expected to be quantitatively accurate very far out in the boundary layer because of the approximations made. Still it is interesting that close to the boundary, where these solutions are most accurate, the shapes of the u and v curves do not change very much with time, only the scale and amplitude. Further out they do, and the ultimate decay to zero predicted here for both u and v is like r^{-4}. The place where u reaches its maximum and v changes sign are fairly close to the outer edge of the primary flow boundary layer. Therefore, we conclude that the secondary flow in general has larger scale, though smaller amplitude, than the primary zonal flow. This feature is also indicated by the steady-state data of Kobashi (1957).

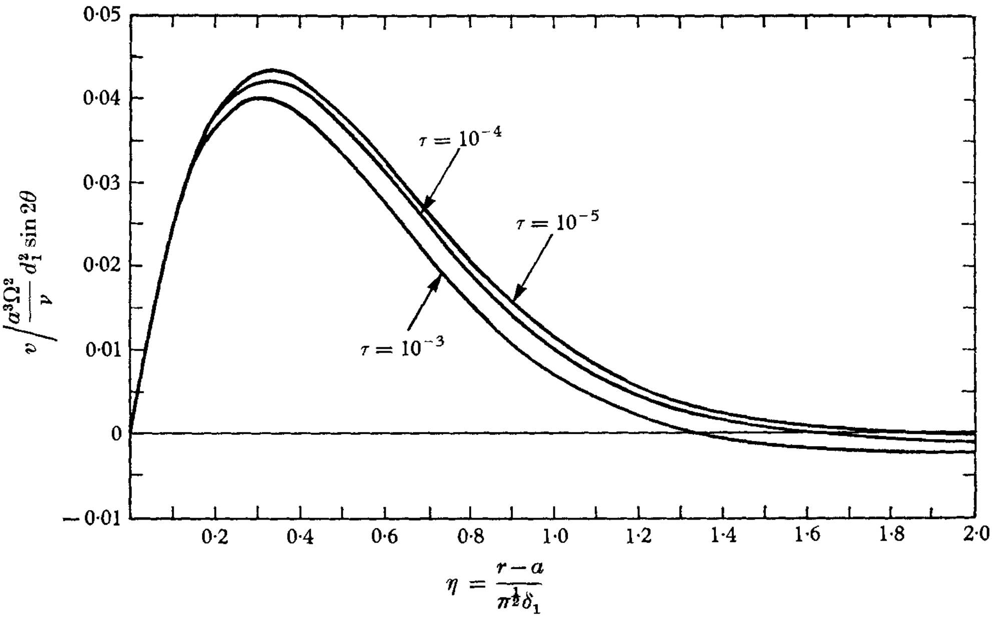

FIGURE 5. The radial dependence of the meridional velocity component for three values of time.

The consistency of the linearization approximation can be checked *post facto* by examining the ratios u/w, v/w which are to be small. Apart from the region far out in the boundary layer, where the present solutions are unreliable anyway, $u \ll v$, so it is sufficient to look at v/w. In the present notation,

$$\frac{v}{w} = 2\frac{a^2\Omega}{\nu}\cos\theta(\pi^{-\frac{1}{2}}x^{-3}d_1^2 G' - 2x^{-4}d_1^3 G)(f)^{-1}. \tag{43}$$

A numerical computation of this function shows that for given Taylor number, $a^2\Omega/\nu$, co-latitude, θ, and radial distance, x, v/w, increases montonically with τ; so we examine the case of $\tau = 10^{-3}$ which is about the largest value for which the present theory is valid. Restricting interest to the region $0 \leqslant \eta \leqslant 1{\cdot}65$, we find that v/w will be small compared to 1 so long as Ωt is small compared to about $4{\cdot}5 \sec\theta$; for example, at $\theta = 60°$, Ωt must be small compared to about 9 radians.

In general, the actual values of u, v, w computed from the present results are felt to be reasonable estimates during no more than the first radian of the sphere's travel, for values of τ up to 10^{-3}, and values of η out to $1{\cdot}65$. Of potentially greater use than the actual numerical values, is the discovery of similarity in the problem.

During the course of this investigation the author benefited from discussions with M. B. Glauert, B. R. Morton, and E. Watson from Manchester University, G. W. Platzman from the University of Chicago, and P. D. Thompson of the National Center for Atmospheric Research. Numerical evaluations were programmed by Mr Loren Wagner of NCAR. The work was supported in part by the Office of Naval Research under contract Nonr 3164(00).

REFERENCES

Carrier, G. 1953 Boundary layer problems in applied mechanics. *Advances in Applied Mechanics*, **3**, 1–19. New York: Academic Press.

Carrier, G. & DiPrima, R. C. 1956 On the torsional oscillations of a solid sphere in a viscous fluid. *J. Appl. Mech.* **23**, 601–605.

Erdélyi, A. 1954 *Tables of Integral Transforms*, vol. I. New York: McGraw-Hill.

Howarth, L. 1951 Note on the boundary layer on a rotating sphere. *Phil. Mag.* **42**, 1308–1315.

Kobashi, Y. 1957 Measurements of boundary layer of a rotating sphere. *J. Sci. Hiroshima Univ.* A, **29**, 149–156.

Kreith, F., Roberts, L. G., Sullivan, J. A. & Sinha, S. N. 1963 Convection heat transfer and flow phenomena of rotating spheres. *Int. J. Heat and Mass Transfer*, **6**, 881–895.

Nigam, S. D. 1951 Rotation of an infinite plane lamina: boundary layer growth: motion started impulsively from rest. *Quart. Appl. Math.* **9**, 89–91.

Nigam, S. D. 1954 Note on the boundary layer on a rotating sphere. *ZAMP*, **5**, 151–155.

Schlichting, H. 1960 *Boundary Layer Theory*, 4th ed. New York: McGraw-Hill.

Stewartson, K. 1958 On rotating laminar boundary layers. *Boundary Layer Research*, pp. 59–71. Symposium Freiburg/BR. Berlin: Springer-Verlag.

Stokes, G. G. 1845 On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids. *Trans. Camb. Phil. Soc.* **8**, 287–319.